14.7. Maximum and minimum values : global extrema

Def Consider a function $f(x,y)$ with domain D.

(1) It has a global maximum at (a,b) if it satisfies

$f(x,y) \le f(a,b)$ on D.

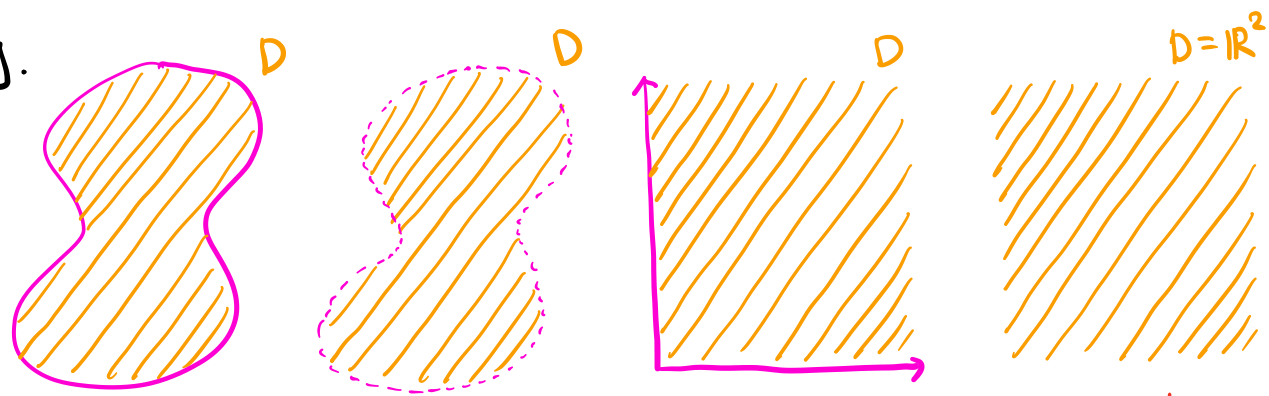
(2) It has a global minimum at (a,b) if it satisfies

$f(x,y) \ge f(a,b)$ on D.

(3) The domain D is

- closed if it contains all of its boundary
- open if it contains none of its boundary
- bounded if it is contained in some circle.

e.g.



closed, bounded | open, bounded | closed, not bounded | open and closed, not bounded

* $\mathbb{R}^2$ has empty boundary, and thus contains all and none of its (empty) boundary.

Recall: A continuous function $f(x)$ on a closed interval must attain global extrema at endpoints or critical points.

☆ **Thm** (Extreme value theorem)

For a continuous function $f(x,y)$ on a closed and bounded domain D, global extrema always exist and can be found as follows:

Step 1. Evaluate $f(x,y)$ at all critical points.

usually difficult Step 2. Find the extrema of $f(x,y)$ on the boundary.

Step 3. Compare all values from Steps 1 and 2.

global maximum = the largest of these values
global minimum = the smallest of these values

Prop For a continuous function on an open domain, critical points are the only possible locations of global extrema.

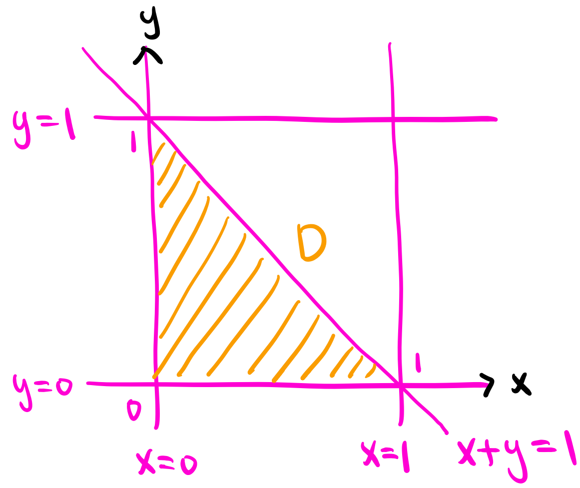
Note For a continuous function on an open domain, there may be no global extrema. However, in Math 215, if a problem asks you to find a global extrema, you can assume that they exist.

Ex Consider the function $f(x,y) = -x^2 - y^2 + 2x + 2y + 12$.

(1) Find all extreme values of $f(x,y)$ on the domain

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x, y \le 1,\ x+y \le 1\}.$$

Sol We can sketch D as follows:



$\Rightarrow$ D is closed and bounded.

We apply the Extreme value theorem.

Step 1. Evaluate $f(x,y)$ at all critical points.

$$\nabla f = (f_x, f_y) = (-2x+2, -2y+2)$$

At critical points, $\nabla f = (0,0)$

$\Rightarrow$ $-2x+2=0$ and $-2y+2=0$

$\Rightarrow$ $x=1$ and $y=1$

However, $(1,1)$ is not in D

$\Rightarrow$ $f(x,y)$ has no critical points on D.

Step 2. Find the extrema of $f(x,y)$ on the boundary.

The boundary consists of three segments.

- The horizontal segment with $y=0$ and $0 \le x \le 1$:

$f(x,y) = f(x,0) = -x^2 + 2x + 12$.

$$\frac{d}{dx}(-x^2+2x+12) = -2x+2$$

$\Rightarrow$ A critical point at $x=1$.

For $0 \le x \le 1$, possible extrema are

$f(0,0) = 12$, $f(1,0) = 13$.

(We consider the endpoints at $x=0,1$ and the critical point at $x=1$)

- The vertical segment with $x=0$ and $0 \le y \le 1$:

$f(x,y) = f(0,y) = -y^2 + 2y + 12$.

$$\frac{d}{dy}(-y^2+2y+12) = -2y+2$$

$\Rightarrow$ A critical point at $y=1$.

For $0 \le y \le 1$, possible extrema are

$f(0,0) = 12$, $f(0,1) = 13$.

(We consider the endpoints at $y=0,1$ and the critical point at $y=1$)

- The diagonal segment with $x+y=1$ and $0 \le x \le 1$:

$$f(x,y) = f(x,1-x) = -x^2-(1-x)^2+2x+2(1-x)+12$$
$$= -2x^2+2x+13.$$

$$\frac{d}{dx}(-2x^2+2x+13) = -4x+2$$

$\Rightarrow$ A critical point at $x = \frac{1}{2}$.

For $0 \le x \le 1$, possible extrema are

$f(0,1) = 13$, $f(\frac{1}{2},\frac{1}{2}) = 13.5$, $f(1,0) = 13$.

(We consider the endpoints at $x=0,1$ and the critical point at $x=1$)

Step 3. Compare all values from steps 1 and 2.

$$\boxed{\begin{array}{l}\text{A global maximum of } 13.5 \text{ at } (\frac{1}{2},\frac{1}{2}) \\ \text{A global minimum of } 12 \text{ at } (0,0)\end{array}}$$

Note In Step 2, it is not enough to only consider the three vertices on the boundary. In fact, the global maximum occurs at a boundary point which is not a vertex.

(2) Find the maximum value of $f(x,y)$ on the xy-plane

<u>Sol</u> The domain $\mathbb{R}^2$ is open.

$\Rightarrow$ A global maximum must be a critical point

$\Rightarrow \nabla f = (0,0) \underset{(1)}{\Rightarrow} (x,y) = (1,1)$

$\Rightarrow$ The maximum value is $f(1,1) = \boxed{14}$

<u>Note</u> $f(x,y)$ attains no minimum values on the xy-plane,

as you get $\lim_{x\to\infty} f(x,0) = \lim_{x\to\infty} (-x^2+2x+12) = -\infty$.